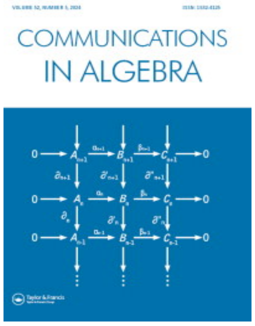

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On groups occurring as absolute centers of finite groups

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ABSTRACT
Given a construction f on groups, we say that a group G is *f-realisable* if there is a group H such that $G \cong f(H)$, and *completely f-realisable* if there is a group H such that $G \cong f(H)$ and every subgroup of G is isomorphic to $f(H_1)$ for some subgroup H_1 of H and vice versa.
Denote by $L(G)$ the absolute center of a group G, that is the set of elements of G fixed by all automorphisms of G. By using the structure of the automorphism group of a ZM-group, in this paper we prove that cyclic groups C_N, $N \in \mathbb{N}^*$, are completely L-realisable.

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1. Introduction

In group theory, there are many constructions f which start from a group H and produce another group $f(H)$. Examples of such group-theoretical constructions are: center, central quotient, derived quotient, Frattini subgroup, Fitting subgroup, Chermak-Delgado subgroup, automorphism group, Schur multiplier, other cohomology groups, and various constructions from permutation groups. For each of these constructions, there is an inverse problem:

$$\text{Given a group } G, \text{ is there a group } H \text{ such that } G \cong f(H)? \tag{1}$$

Several new results related to this problem have been obtained in [1, 2, 10, 11] for $f(H) = D(H)$, the derived subgroup of H. Note that in these papers the group H with the property $G \cong D(H)$ has been called an *integral* of G by analogy with calculus. Moreover, we recall Problem 10.19 in [1] that asks to classify the groups in which all subgroups are integrable.

Other results of the same type are given by [7, 15, 21–23] for $f(H) = \Phi(H)$, the Frattini subgroup of H. In this case, there is a precise characterization of finite groups G for which (1) has solutions, namely

$$G \cong \Phi(H) \text{ for some group } H \text{ if and only if } \mathrm{Inn}(G) \subseteq \Phi(\mathrm{Aut}(G))$$

(see [7]).

The same problem for $f(H) = \mathrm{Aut}(H)$, the automorphism group of H, has been studied in [16, 17]. We also recall the well-known class of *capable groups*, i.e. the groups G such that (1) has solutions for $f(H) = \mathrm{Inn}(H)$, the inner automorphism group of H. Their study was initiated by R. Baer [3] and continued in many other papers (see e.g. [4, 8]).

Inspired by these studies, in [9] we introduced the following two notions. Given a construction f on groups, we say that a group G is

a) *f-realisable* if there is a group H such that $G \cong f(H)$

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and

b) *completely f-realisable* if there is a group H such that:

i) $G \cong f(H)$;
ii) $\forall G_1 \leq G, \exists H_1 \leq H$ such that $G_1 \cong f(H_1)$;
iii) $\forall H_1 \leq H, \exists G_1 \leq G$ such that $f(H_1) \cong G_1$.

We determined completely Aut-realisable groups. We also provided some results about (completely) f-realisable groups for $f = Z, F, M, D, \Phi$, where $Z(H)$, $F(H)$, and $M(H)$ denote the center, the Fitting subgroup and the Chermak-Delgado subgroup of the group H, respectively.

The current paper deals only with finite groups. We will continue the above study for $f = L$, where $L(H)$ is the absolute center of the group H, i.e.

$$L(H) = \{h \in H \mid \alpha(h) = h, \forall \alpha \in \mathrm{Aut}(H)\}.$$

It has been introduced by Hegarty [12], together with the autocommutator subgroup of the group H:

$$K(H) = \langle h^{-1}\alpha(h) \mid h \in H, \alpha \in \mathrm{Aut}(H)\rangle.$$

Hegarty proved an analogue of Schur's theorem for the absolute center and the autocommutator subgroup, namely that if H is a group such that $H/L(H)$ is finite, then so are $K(H)$ and $\mathrm{Aut}(H)$. The converse of this result is also true, as it has been shown in [13].

Clearly, the absolute center of a group is abelian, since $L(H) \subseteq Z(H)$ for all groups H. It was determined for several classes of p-groups, such as for abelian p-groups in [6], for minimal non-abelian p-groups in [18] or for p-groups of maximal class in [19]. Note that in all these cases we have

$$L(H) \cong C_p^k \text{ for certain } k = 0, 1, 2.$$

Also, absolute centers of direct products of such groups of coprime orders will be direct products of elementary abelian p-groups of rank ≤ 2, according to Lemma 2.1 of [18]. This leads to the following natural question:

$$\text{Which (abelian) groups occur as absolute centers of finite groups?} \tag{2}$$

In what follows, we will show that finite cyclic groups satisfy this property. More precisely, the following more powerful result holds.

Theorem 1.1. *All finite cyclic groups are completely L-realisable.*

The proof of Theorem 1.1 is based on computing the absolute center of a ZM-group, that is a finite group all of whose Sylow subgroups are cyclic. By [14], such a group is of type

$$\mathrm{ZM}(m, n, r) = \langle a, b \mid a^m = b^n = 1,\ b^{-1}ab = a^r\rangle,$$

where the triple (m, n, r) satisfies the conditions

$$\gcd(m, n) = \gcd(m, r-1) = 1 \text{ and } r^n \equiv 1 \pmod{m}.$$

Note that $|\mathrm{ZM}(m, n, r)| = mn$, $\mathrm{ZM}(m, n, r)' = \langle a\rangle$ and $Z(\mathrm{ZM}(m, n, r)) = \langle b^d\rangle$, where d is the multiplicative order of r modulo m, i.e.

$$d = o_m(r) = \min\{k \in \mathbb{N}^* \mid r^k \equiv 1 \pmod{m}\}.$$

Finally, we mention that we were unable to give a complete answer to question (2), but, based on the above results, the following conjecture seems reasonable:

Conjecture. *All finite abelian groups are completely L-realisable.*

Most of our notation is standard and will usually not be repeated here. Elementary notions and results on groups can be found in [14]. For subgroup lattice concepts we refer the reader to [20].

2. Automorphisms of ZM-groups

The general form of automorphisms of a metacyclic group has been determined in the main result of [5]. In our case, we get:

Theorem 2.1. *Each automorphism of* ZM(m, n, r) *is given by*

$$b^u a^v \mapsto b^{yu} a^{x_1 v + x_2 [u]_r},\ u, v \geq 0,$$

for a unique triple of integers (x_1, x_2, y) *such that*

$$0 \leq x_1, x_2 < m,\ (x_1, m) = 1,\ 0 \leq y < n \text{ and } y \equiv 1 \pmod{d},$$

where

$$[u]_r = \begin{cases} 1 + r + \cdots + r^{u-1}, & u > 0 \\ 0, & u = 0. \end{cases}$$

In particular, we have

$$|\mathrm{Aut}(\mathrm{ZM}(m, n, r))| = m\varphi(m)\frac{n}{d}.$$

Since Inn(ZM(m, n, r)) is of order md, Theorem 2.1 leads to the following corollary.

Corollary 2.2. *We have*

$$|\mathrm{Out}(\mathrm{ZM}(m, n, r))| = \frac{\varphi(m)n}{d^2}$$

and

$$\mathrm{ZM}(m, n, r) \text{ is a complete group } \Leftrightarrow \varphi(m) = n = d.$$ [1]

The form of central automorphisms and of IA-automorphisms of ZM(m, n, r) can be also inferred from Theorem 2.1.

Corollary 2.3. *Each central automorphism of* ZM(m, n, r) *is given by*

$$b^u a^v \mapsto b^{yu} a^v,\ u, v \geq 0,$$

for a unique integer y *such that*

$$0 \leq y < n \text{ and } y \equiv 1 \pmod{d}$$

and each IA-automorphism of ZM(m, n, r) *is given by*

$$b^u a^v \mapsto b^u a^{x_1 v + x_2 [u]_r},\ u, v \geq 0,$$

for a unique pair of integers (x_1, x_2) *such that*

$$0 \leq x_1, x_2 < m \text{ and } (x_1, m) = 1.$$

In particular, we have

$$|\mathrm{Aut}_{\mathrm{C}}(\mathrm{ZM}(m, n, r))| = \frac{n}{d} \text{ and } |\mathrm{Aut}_{\mathrm{IA}}(\mathrm{ZM}(m, n, r))| = m\varphi(m).$$

Next, we determine the absolute center of ZM(m, n, r).

[1] We note that in this case m is a cyclic number.

Theorem 2.4. *We have*

$$L(\mathrm{ZM}(m,n,r)) = \langle b^{de}\rangle,$$

where

$$e = \min\{s \in \mathbb{N}^* \mid d^2 s \equiv 0 \pmod n\}.$$

Moreover

$$|L(\mathrm{ZM}(m,n,r))| = \frac{n}{(de,n)} = \frac{\frac{n}{d}}{(e,\frac{n}{d})}.$$

Proof. Since $L(\mathrm{ZM}(m,n,r)) \subseteq Z(\mathrm{ZM}(m,n,r))$, it follows that

$$L(\mathrm{ZM}(m,n,r)) = \langle b^{de}\rangle \text{ for some } e \in \mathbb{N}^*.$$

By Theorem 2.1, the condition $b^{de} \in L(\mathrm{ZM}(m,n,r))$ means

$$b^{yde}a^{x_2[de]_r} = b^{de},$$

that is

$$n \mid de(y-1) \text{ and } m \mid x_2[de]_r \tag{3}$$

for all pairs of integers (x_2, y) satisfying $0 \le x_2 < m$, $0 \le y < n$ and $y \equiv 1 \pmod d$. Clearly, the relations (3) hold if and only if they hold for $x_2 = 1$ and $y = d$, that is

$$n \mid d^2 e \text{ and } m \mid [de]_r.$$

Since $m \mid [de]_r$ is true for all $e \in \mathbb{N}^*$, we must have only $n \mid d^2 e$ and the proof is completed by the remark that e must be chosen minimal with respect to this condition. $\square$

For example, for $\mathrm{ZM}(5,16,2) = SmallGroup(80,3)$ we get $d = 4$, $e = 1$ and so $L(\mathrm{ZM}(5,16,2)) = Z(\mathrm{ZM}(5,16,2)) \cong C_4$, while for $\mathrm{ZM}(5,48,2) = SmallGroup(240,5)$ we get $d = 4$, $e = 3$ and so $L(\mathrm{ZM}(5,48,2)) \cong C_4$. Note that in the second case we have $L(\mathrm{ZM}(5,48,2)) \neq Z(\mathrm{ZM}(5,48,2)) \cong C_{12}$.[2]

3. Proof of the main result

First of all, we will show that $L(\mathrm{ZM}(m,n,r))$ can be any cyclic group C_N when $N = q^\alpha$ is a prime power. By Theorem 2.4, we must find a triple of positive integers (m,n,r) such that

$$\begin{cases} \gcd(m,n) = \gcd(m,r-1) = 1 \\ r^n \equiv 1 \pmod m \\ \frac{n}{(de,n)} = N, \end{cases} \tag{4}$$

where

$$\begin{cases} d = o_m(r) \\ e = \min\{s \in \mathbb{N}^* \mid d^2 s \equiv 0 \pmod n\}. \end{cases}$$

We start with the following auxiliary result.

Lemma 3.1. *There exist a prime p and a positive integer r such that $o_p(r) = q^\alpha$.*

Proof. By Dirichlet's theorem, the arithmetic progression $1 + tq^\alpha$, $t \in \mathbb{N}$, contains an infinite number of primes. Let p be one of them. Then q^α divides $p-1$. Since the multiplicative group of integers modulo p is cyclic of order $p-1$, it contains elements of order q^α. Thus there is $r \in \mathbb{N}^*$ such that $o_p(r) = q^\alpha$, as desired. $\square$

[2]More precisely, since $\mathrm{ZM}(5,48,2) \cong C_3 \times \mathrm{ZM}(5,16,2)$, we have $L(\mathrm{ZM}(5,48,2)) \cong L(C_3) \times L(\mathrm{ZM}(5,16,2)) \cong 1 \times C_4 \cong C_4$ and $Z(\mathrm{ZM}(5,48,2)) \cong Z(C_3) \times Z(\mathrm{ZM}(5,16,2)) \cong C_3 \times C_4 \cong C_{12}$.

Clearly, the triple $(p, q^{2\alpha}, r)$, where p and r are given by Lemma 3.1, satisfies the conditions (4), that is we have

$$L(\mathrm{ZM}(p, q^{2\alpha}, r)) \cong C_{q^\alpha}.$$

For the proof of Theorem 1.1, we also need the following corollary.

Corollary 3.2. *Let P be the set of all primes. Then for any $q_1, \ldots, q_k \in P$ and any $\alpha_1, \ldots, \alpha_k \in \mathbb{N}^*$, there exist distinct primes $p_1, \ldots, p_k \in P \setminus \{q_1, \ldots, q_k\}$ and positive integers $r_1, \ldots, r_k$ such that $o_{p_i}(r_i) = q_i^{\alpha_i}$, $i = 1, \ldots, k$.*

Proof. We apply Lemma 3.1 for all $i = 1, \ldots, k$. It suffices to observe that the primes $p_1, \ldots, p_k$ can be chosen to be distinct, and different from $q_1, \ldots, q_k$, by Dirichlet's Theorem. □

We are now able to prove our main result.

Proof of Theorem 1.1. Let $N \geq 2$ be an integer and $N = q_1^{\alpha_1} \cdots q_k^{\alpha_k}$ be the decomposition of N as a product of prime factors. We choose $p_1, \ldots, p_k$ and $r_1, \ldots, r_k$ as in Corollary 3.2, and let

$$H_i = \mathrm{ZM}(p_i, q_i^{2\alpha_i}, r_i),\ i = 1, \ldots, k.$$

Then, for each i, we have

$$L(H_i) \cong C_{q_i^{\alpha_i}}.$$

Let $H = H_1 \times \cdots \times H_k$. Since the groups H_i, $i = 1, \ldots, k$, are of coprime orders, we infer that they are characteristic in H and therefore

$$L(H) \cong L(H_1) \times \cdots \times L(H_k) \cong C_{q_1^{\alpha_1}} \times \cdots \times C_{q_k^{\alpha_k}} \cong C_N,$$

i.e. C_N is L-realisable.

Obviously, every subgroup G_1 of C_N is of type C_{N_1}, where $N_1 = q_1^{\beta_1} \cdots q_k^{\beta_k}$ and $0 \leq \beta_i \leq \alpha_i$, $i = 1, \ldots, k$. It is easy to see that each group H_i contains a normal subgroup $S_i \cong \mathrm{ZM}(p_i, q_i^{\alpha_i+\beta_i}, r_i)$. Since all normal subgroups of a ZM-group are characteristic, it follows that S_i is characteristic in H_i, and so in H. Moreover, we have $L(S_i) \cong C_{q_i^{\beta_i}}$. Then the subgroup $S = S_1 \times \cdots \times S_k$ of H satisfies $L(S) \cong G_1$.

Conversely, we observe that every subgroup T of H is a direct product $T_1 \times \cdots \times T_k$, where $T_i \leq H_i$, $i = 1, \ldots, k$. Then each T_i is either cyclic or of type $\mathrm{ZM}(p_i, q_i^{\gamma_i}, r_i)$ with $\alpha_i \leq \gamma_i \leq 2\alpha_i$. In the second case, we get

$$L(T_i) \leq Z(T_i) \cong C_{q_i^{\gamma_i - \alpha_i}}.$$

Consequently, $L(T_i)$ is either trivial or isomorphic to a subgroup of $C_{q_i^{\alpha_i}}$. It follows that $L(T) \cong L(T_1) \times \cdots \times L(T_k)$ is isomorphic to subgroup of $C_{q_1^{\alpha_1}} \times \cdots \times C_{q_k^{\alpha_k}} \cong C_N$.

This completes the proof. □

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